GENERALIZATIONS IN THE THEORY OF ONE-DIMENSIONAL RADIATIVE HEAT TRANSFER

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The profound generality of transfer processes is evident in the physical analogies which have been successfully included in a number of theories of heat transfer [1–11]. The basic content of these analogies, which have been sufficiently described in specific investigations (rarefied media [4–10], turbulent transfer [11]) consists in a generalization of the transfer process model. The differences in principle in the nature of the heat transfer agent (photons or particles) turns out not to be very important. The basis of the modelling is the kinetic Boltzmann equation, which, in this simplified interpretation [4], is analogous to the equation of radiative energy transfer.

When applied to the particular case of a steady field in a nonscattering homogeneous medium, the kinetic Boltzmann equation may be written as follows:

$$df(M, S) / ds = = k(M, S)\{-f(M, S) + \varepsilon_0(M, S)\}. \qquad (1)$$

Here $f(M, S)$ has the meaning of a specific intensity of "radiation" of photons (or neutrons) or of particles (molecules, and their combinations), depending on the nature of the transfer processes, at the point M and in the direction S; when the energy of the elementary carriers and the velocity of their propagation is taken into account, a direct relation is established between $f(M, S_0)$ and the distribtuion function [12]; $\varepsilon_0(M, S)$ is an equilibrium function representing emission and production of photons or particles in an element of volume at the point M in direction S, and associated with the elementary interaction processes; $k(M, S)$ is the attenuation (extinction) coefficient, associated with the interaction, and including the scattering and absorption coefficients, in general.

The transfer equation (1), taking account of the boundary conditions, may be transformed, by a formal integration, into an equation which is a solution with respect to $f(M, S)$:

$$f(M, S) = f(M_0, S_0)e^{-h(M, S)} + \int_{S_0}^{S} \varepsilon_0(N, S)\, e^{(-h(N, S)+h(M, S))}\, dh$$
$$\left(h(M, S) = \int_{S_0}^{S} k(P, S)\, dS\right),$$
$$dh = k(N, S)\, dS. \qquad (2)$$

Here $f(M_0, S_0)$ is the boundary value of the specific intensity, and $h(M, S)$ is the optical density of the medium, or the number of mean free paths of the carrier fitting into the ray $S_0 - S$.

The transfer processes are characterised by flux densities which are given various physical interpretations, depending on the nature of the investigation.

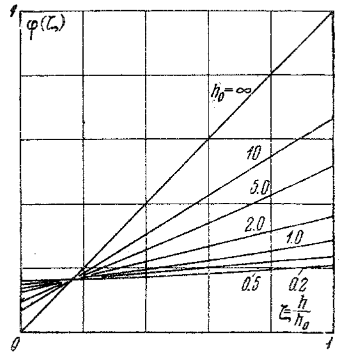

Fig. 1

The density of the resultant energy transfer across an imaginary plane in space is given by the relation.

$$E(M, S) = \pi \int_{4\pi} f(N, S)\, Q(M, S, N)\, dF_N. \qquad (3)$$

Here $Q(M, S, N)dF_N$ is the spatial orientation function for the elementary transfer agents at fixed M and current N points of contact with respect to the chosen direction S.

Examination of one-dimensional transfer processes reduces to investigation of the energy equation expressing the resultant volume transfer. From analogy with the expression for the resultant heat transfer in a plane layer of a gray, nonscattering medium (the walls emit diffusely) [13], the above equation may be written in the form

$$-\frac{1}{\pi}\eta(h) = \frac{1}{\pi}\frac{dE(h)}{dh} =$$
$$= 4\varepsilon_0(h) - 2\sigma_1(h)f(h_1) - 2\sigma_2(h)f(h_2) -$$
$$-2\int_0^{h_0} \varepsilon_0(\zeta)\, G(h, \zeta)\, d\zeta. \qquad (4)$$

Here $G(h, \zeta)$ is a functional describing the volume transfer, including the effects of reflection at the boundaries; $\sigma_i(h)f(h_i)$ is the emission of the boundary surfaces, attenuated by the intervening medium ($i = 1, 2$), and by reflections at the boundaries on the way to h.

For a given value of $\eta(h)$, Eq. (4) allows us to determine the equilibrium distribution function $\varepsilon_0(\zeta)$ over the layer, and therefore, the corresponding distributions of temperature, velocity, and so on.

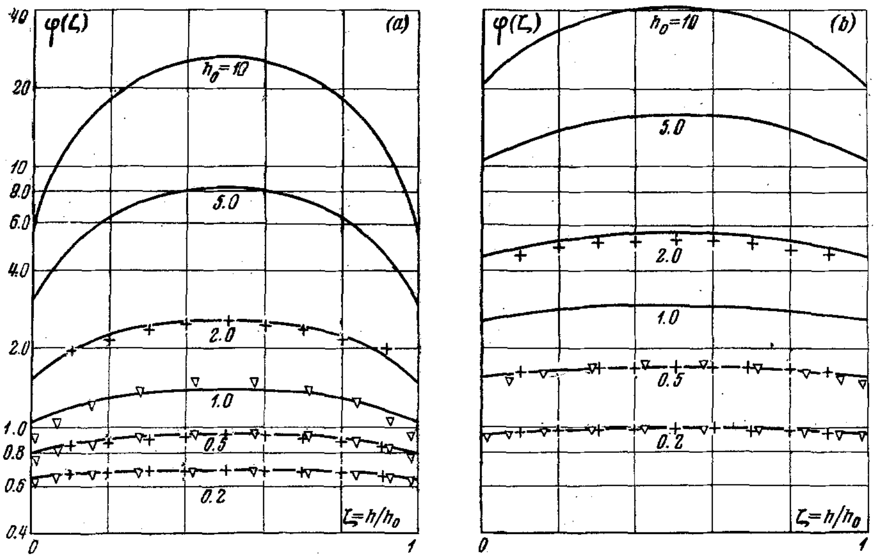

Fig. 2

The boundary influence functions $\sigma_i(h)$ and the functional $G(h, \zeta)$ are determined from the relations [13]

$$\sigma_1(h) = A_1 \frac{K_2(h) + 2(1 - A_2) K_3(h_0) K_2(h_0 - h)}{1 - 4(1 - A_1)(1 - A_2) K_3^2(h_0)},$$

$$\sigma_2(h) = A_2 \frac{K_2(h_0 - h) + 2(1 - A_1) K_3(h_0) K_2(h)}{1 - 4(1 - A_1)(1 - A_2) K_3^2(h_0)},$$

$$G(h, \zeta) = K_1 |h - \zeta| + 2\sigma_1(h) \frac{1 - A_1}{A_1} K_2(\zeta) + $$

$$+ 2\sigma_2(h) \frac{1 - A_2}{A_2} K_2(h_0 - \zeta),$$

$$K_n(x) = \int_0^1 \exp\left(-\frac{x}{\mu}\right) \mu^{n-1} \frac{d\mu}{\mu}. \qquad (5)$$

Here A_i are the generalized accommodation coefficients for the boundary surfaces [12] (either the emittance or the absorptance during emission).

If a solenoidal field $\eta \equiv 0$ for the carriers is being analyzed, the problem of determining the distribution of functions $\varepsilon_0(h)$ transforms to solution of the integral equation

$$\varphi(h) = \frac{1}{2}\sigma_2(h) + \frac{1}{2}\int_0^{h_0} G(h, \zeta)\varphi(\zeta)\, d\zeta \qquad (6)$$

written with respect to the function

$$\varphi(h) = \frac{\varepsilon_0(h) - f(h_1)}{f(h_2) - f(h_1)} = \frac{E_0(h) - E_{0,1}}{E_{0,2} - E_{0,1}},$$

which is a dimensionless analog of the equilibrium function $\varepsilon_0(h)$ (equilibrium emission, temperature, velocity).

It is therefore possible to describe transfer processes in a plane layer due to boundary perturbations (the walls have different temperatures or different velocities of motion). The generalized accommodation coefficient is the parameter which characterizes the perturbations. With reference to the case $\sigma_i(h) \equiv 1$, Eq. (6) was used earlier in analyzing internal friction in rarefied gases [1], and then, later, in investigation of Couette flow in a rarefied medium [10].

If the boundary conditions are symmetrical ($f(h_1) = f(h_2)$, $A_1 = A_2$), then (4) transforms into the integral equation

$$\varphi(h) = \frac{1}{2} + \frac{1}{2}\int_0^{h_0} G(h, \zeta)\varphi(\zeta)\, d\zeta$$

$$G(h, \zeta) = K_1 |h - \zeta| + \sigma_1(h) K_2(\zeta) +$$

$$+ \sigma_2(h) K_2(h_0 - \zeta)$$

$$\left(\varphi(h) = \frac{\varepsilon_0(h) - f(h_1)}{{}^1\!/_2\,\eta\pi} = \frac{\varepsilon_0(h) - f(h_2)}{{}^1\!/_2\,\eta\pi} = \frac{E_0(h) - E_{0,i}}{{}^1\!/_2\,\eta}\right). \qquad (7)$$

Attention should be drawn to the absence in (7) of free terms describing the influence of the boundaries in explicit form. This is associated with the fact that the only source of perturbations in this case is uniformly distributed in this case throughout the entire medium.

Equation (7) allows us to analyze the temperature distribution in a nonconducting, emitting medium, containing uniformly distributed heat sources. It evidently does not allow us to establish the distribution of velocity of flow of a rarefied medium moving in a plane channel with slip.

The dynamic pressure head in the channel* plays the role of sources of perturbations. It is evident that the distribution functions $\varphi(h)$ or ($\varepsilon_0(h)$) undergo a break in continuity in the regions close to the walls, regions whose dimensions are determined by the molecular or optical thicknesses.

The inclusion of appreciable asymmetry into the emission characteristics of the boundaries ($A_2 = 1$, $A_1 \neq 1$) leads to transformation of the influence function $G(h, \zeta)$ in (7) to the form

$$G(h, \zeta) = K_1 |h - \zeta| + 2R_1 K_2(h) K_2(\zeta). \qquad (8)$$

This implies a corresponding asymmetry in the distribution of $\varphi(h)$. A special case of asymmetry inside one of the boundary surfaces is an absolute sink ($f(h_2) \equiv 0$, $A_2 \equiv 1.0$).

*It is clear that (7) also describes the distribution of velocity in the turbulent core of a plane free jet, inside which the pressure, which is the source of perturbations, remains constant.

Then (4) takes the form

$$\varphi(h) = \frac{1}{2}\sigma_1(h) + \frac{1}{2}\int_0^{h_0} G(h, \zeta)\varphi(\zeta)\,d\zeta,$$

$$\sigma_1(h) = 1 + A_1K_2(h)\varphi_1, \quad \varphi(h) = \frac{\varepsilon_0(h)}{{}^1/_2\,\eta\pi},$$

$$\varphi_1 = \frac{f(h_1)}{{}^1/_2\,\eta\pi} = \frac{E_{0,1}}{{}^1/_2\,\eta}.$$

Equation (9) describes processes of heat transfer by radiation in a plane layer with heat generation, one of whose surfaces is absolutely black is at a temperature of absolute zero. The same equation evidently describes the distribution of temperature and velocity in a layer of a rarefied stream flowing over a flat plate with slip.

It may be seen that the processes examined above are described by a Fredholm integral equation of the second kind, which, in general, may be written as:

$$\varphi(h) = \frac{1}{2}\sigma(h) + \frac{1}{2}\int_0^{h_0} G(h, \zeta)\varphi(\zeta)\,d\zeta. \qquad (10)$$

The kernel $G(h, \zeta)$ of the integral equation has a peculiarity as $\zeta \to h$ in that the value of the exponential integral $K_1|h - \zeta|$ contained in it has a break in continuity of logarithmic character. In general, all the Fredholm theorems [15] are valid for a kernel of this type.

However, the specific form (10) associated with numerical solution gives a large error. It is expedient to eliminate the singular point of the functional region $G(h \sim \zeta)$ by writing (10) in the form

$$\varphi(h) = \beta(h)\sigma(h) + \beta(h)\int_0^{h_0} G(h, \zeta)[\varphi(\zeta) - \varphi(h)]\,d\zeta$$

$$\beta(h) = \left(2\left(1 - \frac{1}{2}\int_0^{h_0} G(h, \zeta)\,d\zeta\right)\right)^{-1}. \qquad (11)$$

In the region of small values of the influence function $G(h, \zeta)$, which corresponds to small optical depth or large mean free path, the role of the integral in the functional equation (11), which expresses the volume interaction, is negligible, and the solution (11) may be represented by the simplified relation

$$\varphi(h) \sim \beta(h)\sigma(h). \qquad (12)$$

The solution of (11) relating to the specific cases of investigation of radiative heat transfer may be performed according to the usual iteration scheme when there is a discrete representation of the interval $(0, h_0)$ in the Gauss technique (10 points), bringing in Newton's method. The number of iterations, as a rule, does not exceed two. (All the computations described here were performed on the 20 VTs SO AN SSSR electronic computer.)

1. A plane layer of gray nonscattering medium (without heat generation) with asymmetric boundary conditions. The results of solution of the integral equation (6) are shown in Fig. 1, and are in good agreement with the results of [16]; they show the effect of the optical properties of the boundary surfaces A_i, and also of the optical thickness h_0 of the medium on the distribution of the dimensionless equilibrium emission $\varphi(\zeta)$ in the layer. The case analyzed here has appreciably different values $A_1 = 0.7$ and $A_2 = 0.2$ with reference to the several characteristic values of the optical thickness h_0 of the medium. The solution of (6) may be approximated well by the linear equation

$$\varphi(h) = \varphi(0) + (\varphi(h_0) - \varphi(0))\,h/h_0, \qquad (13)$$

where $\varphi(0)$ and $\varphi(h_0)$ (the values of $\varphi(h)$ in the region near the walls) are determined comparatively simply [13] from simultaneous examination of (6) and (12). In the special case when the boundary surfaces are absolutely black

$$\varphi(0) = ({}^1/_2 - K_3(h_0))\,(1 + h_0(1 - K_2(h_0)) - 2K_3(h_0))^{-1},$$

$$\varphi(h_0) = 1 - \varphi(0). \qquad (14)$$

The results of the solution with (13) included are in satisfactory agreement with those of the strict formulation of the problem (Fig. 1) examined earlier with reference to the case $A_1 = 0.7$ and $A_2 = 0.2$ (Fig. 3 of [13]).

This agreement is evidence that the linear approximation to the rigorous solution is applicable even in the general case when the boundary conditions are asymmetric as regards to optical properties of the surfaces. The results obtained have an immediate relevance to investigations of the distribution of velocities and temperatures in a plane layer of rarefied gas formed by two infinite planes, of which one (the second) is in motion (Couette flow) [1, 10].

In addition, being very graphic in a physical sense, the results give appropriate information in constructing approximate solutions.

In weakly absorbing media ($h_0 < 0.5$) we may use relation (12), which takes the form, with reference to the case under analysis,

$$\varphi(h) = \sigma_2(h)\Big\{K_2(h) + K_2(h_0 - h) - (1 - 2K_3(h_0))\left(\sigma_1(h)\frac{1 - A_1}{A_1} + \sigma_2(h)\frac{1 - A_2}{A_2}\right)\Big\}^{-1}. \qquad (15)$$

For the case with absolutely black boundary surfaces

$$\varphi(h) = K_2(h_0 - h)\,(K_2(h) + K_2(h_0 - h))^{-1}. \qquad (16)$$

Calculations of the distribution $\varphi(h)$ by formula (15) with reference to the case $A_1 = 0.7$, $A_2 = 0.2$, $h_0 = 0.2$ gave very good results.

2. Plane layer of medium with uniformly distributed heat sources (symmetric boundary conditions).* Solution of (7), as represented in (10), was carried out for a large range of values of h_0 and $R = 1 - A$. The corresponding numerical results are shown in Fig. 2, which shows the distribution of values of equilibrium radiation

$$\varphi(\zeta) = \frac{E_0(\zeta) - E_{0.1}}{E_{0.2} - E_{0.1}}$$

as a function of thickness of the layer of gray medium of different optical thickness, when the boundary surfaces reflect radiation diffusely with different intensities (a) $R = 0$, (b) $R = 0.6$. Point 1 was obtained from (17), and point 2 from (23). It may be seen that with an increase of h_0 the degree of nonequilibrium in the values of $\varphi(h)$ first increases, and then decreases again. The increase of the reflectance R in the above sense always has a stabilizing influence. Similar results were obtained in [17] by the Monte Carlo method.

The solution of the integral equation (7) may be approximated well by a parabola of the form

$$\varphi(h) = \varphi(0) + 4\frac{\varphi({}^1/_2 h_0) - \varphi(0)}{h_0}\,h\left(1 - \frac{h}{h_0}\right). \qquad (17)$$

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*The heat sources (or sinks) indicate that there is some heat generation per unit "optical volume" $\eta(h) = dE(h)/dh$.

Here $\varphi(0)$ and $\varphi(h_0/2)$ are dimensionless values of equilibrium radiation in the near-wall and near-axis regions, respectively, of the plane layer.

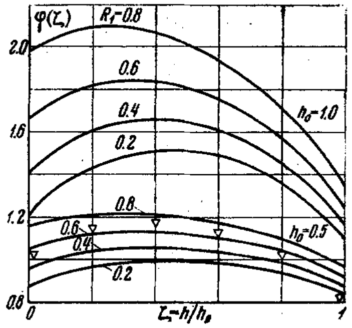

Fig. 3

The values of $\varphi(0)$ and $\varphi(h_0/2)$ are determined from simultaneous examination of (7) and (17). In general, the respective computation equations are awkward, and we therefore restrict attention to the special case of absolutely black boundary surfaces.

To determine $\varphi(0)$ we shall start from solution of the integral equation (7), which allows us to obtain the rigorous relation

$$\varphi(0) = \frac{1}{2} + \frac{1}{4}\int_0^{h_0} \Gamma_1(h)\, dh\,,$$

$$\Gamma_1(h) = K_1(h) + \frac{1}{2}\int_0^{h_0} K_1|h-\zeta|\,\Gamma_1(\zeta)\, d\zeta\,. \tag{18}$$

Using the approximation

$$\int_0^{h_0} \Gamma_1(h)\, dh \approx h_0 \frac{1-K_2(h_0)}{{}^1/_2 - K_3(h_0)} \tag{19}$$

obtained earlier in [13], we have

$$\varphi(0) = \frac{1}{2}\left(1 + h_0 \frac{1-K_2(h_0)}{1-2K_3(h_0)}\right). \tag{20}$$

Using the approximation (17) in the integral equation (7) (for $R \equiv 0$), we have

$$\varphi({}^1/_2 h_0) = \varphi(0) + ({}^1/_2 h_0^2 ({}^1/_2 - \varphi(0) K_2({}^1/_2 h_0)) \times$$
$$\times ({}^2/_3 - h_0 K_3({}^1/_2 h_0) - 2K_4({}^1/_2 h_0))^{-1} \tag{21}$$

The results of the calculations carried out using (17), (21), and (22) are in good agreement with the corresponding values presented in Fig. 2a in the rigorous examination.

With increase of the reflectance of the boundaries (Fig. 2b), the error in the calculation increases, but, however, does not exceed 7% for R = 0.6 and h_0 = 2.0.

The approximate solution (12), which exists for optically weak media (h_0 < 0.5), may be written in the form

$$\varphi(h) = \{K_2(h) + K_2(h_0 - h) -$$
$$- (1 - 2K_3(h_0))\,(\sigma_1(h) + \sigma_2(h))\}^{-1}\,. \tag{22}$$

The results of calculations according to (22) are shown in Fig. 2. It may be seen that the error in calculation for $h_0 \leq 0.5$ does not exceed 10%, even in the general case of a gray surfaces.

3. A plane layer of a medium with uniformly distributed heat sources (asymmetric case). a) One of the boundary surfaces is absolutely black ($A_2 \equiv 1.0$).

The results of solution of the integral equation (7), taking into account (8), as shown in Fig. 3, are evidence of the important influence of asymmetry of optical properties on the distribution of dimensionless equilibrium radiation; with increase in the optical density h_0 this influence is especially noticeable for the reason that the distribution

$$\varphi(\zeta) = \frac{E_0(\zeta) - E_{0.1}}{{}^1/_2 \eta} \tag{23}$$

over the thickness of the layer turns out here to be connected with the specific nature of the boundary conditions to an incomparably great extent.

With decrease of $R_1 = 1 - A_1$, the distribution of $\varphi(\xi)$ becomes more symmetrical, and in the limit, when $R_1 \equiv 0$, symmetry of the distribution of $\varphi(\zeta)$ is restored (the special case of solution of (7), shown in Fig. 2a).

In optically thin layers (h_0 < 0.5), the solution of (7) is approximated satisfactorily by (12), which may be written in the form

$$\varphi(h) = \{K_2(h) +$$
$$+ K_2(h_0 - h) - R_1 K_2(h)\,(1 - 2K_3(h_0))\}^{-1}\,. \tag{24}$$

The results of calculations according to (24) are shown in Fig. 3 for h_0 = 0.5 and R_1 = 0.6.

b) One of the boundary surfaces is assumed to be at the temperature absolute zero ($E_{0.2} \equiv 0$, $A_2 \equiv 1$). The results of numerical solution of the integral equation (9) for the case φ_1 = 1.0 for a wide range of R_1 and h_0 are shown in Fig. 4 in the form

$$\varphi(\zeta) - \varphi_1 = F(\zeta) = \frac{E_0(\zeta) - E_{0.1}}{{}^1/_2 \eta}.$$

In Fig. 4 curves 1, 2, 3, 4, and 5 correspond to the values R_1 = = 0, 0.2, 0.4, 0.6, 0.8.

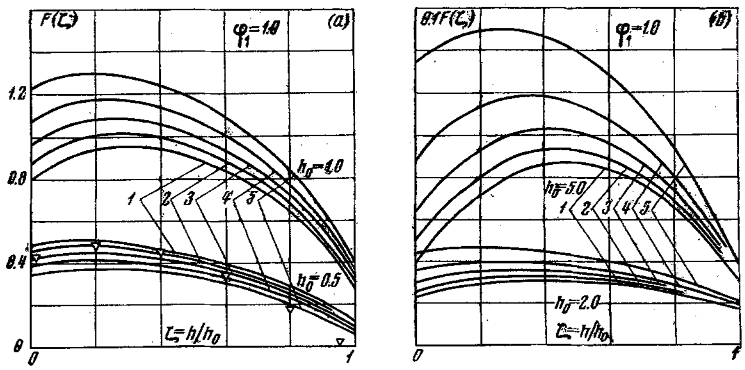

Fig. 4

From comparison of the results of calculations shown in Fig. 4a with the analogous results in Fig. 3, it is very evident that there is a decrease in the general level of $\varphi(\zeta)$, due to the peculiar "luminescence" on the part of the wall at absolute zero.

For a specific ratio of the chosen parameters as regards optical density of the layer h_0, and also for dimensionless equilibrium radiation of the actual wall $\varphi_1 = E_{0,1}/2\eta$, the dimensionless equilibrium radiation $\varphi(\zeta)$ increase with increase of the reflectance of the wall R_1 (Fig. 4a for $h_0 = 0.5$). This is due to the dominant role of the emittance of the wall in the distribution of $\varphi(\zeta)$. It is clear that with increase in the optical density h_0, the role becomes weaker, and to an increasing extent the internal processes begin to show up, in a medium containing a uniform distribution of heat sources (Fig. 4b).

In optically thin media ($h_0 < 0.5$), we may use the approximate expression (12), which in this case may be written as

$$\varphi(h) \approx (1 + (1 - R_1) K_2(h) \varphi_1) (K_2(h) \times \times (1 - R_1 (1 - 2 K_3(h_0))) + K_2(h_0 - h))^{-1}. \qquad (25)$$

Calculations according to (25) relative to the case $h_0 = 0.5$, $R_1 = 0.6$, and $\varphi_1 = 1.0$ are shown in Fig. 4a. The maximum error of calculations of this kind also does not exceed 10%. For this reason the approximation (12) and its special forms represented by (15), (23), (24), and (25) should be used in the range of optical thickness confined to $h_0 < 0.5$.

A more effective formula, in the sense of wide range of optical density, is represented by the approximate solutions (13) and (17), based as they are on additional knowledge of the expected nature of the distribution $\varphi(\zeta)$. The generalized physical analogies examined approximately in this paper may be very useful in this sense.

The topics touched upon are examples of the simplest cases in the approximate development of the general theory of transfer.

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